Equidistributed

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The content is from [1, Chapter 4, Section 2].

Definition 1. *A sequence of number* $\{\xi_n\}_{n\in\mathbb{N}} \subset [0,1)$ *is said to be equidistributed if for any interval* $(a,b) \subset [0,1)$*,*

$$\lim_{N\to\infty} \frac{\#\{1 \le n \le N : \xi_n \in (a,b)\}}{N} = b - a.$$

where the $\#A$ *denotes the cardinality of the finite set* A*.*

Definition 2. *Let* $[x]$ *denote the greatest integer less than or equal to* x *and call* $[x]$ *the integer part of* x*. Let* $\langle x\rangle := x - [x]$ *and call* $\langle x\rangle$ *the fractional part of* x*.*

Theorem 1 (Weyl's criterion)**.** *A sequence of number* $\{\xi_n\}_{n\in\mathbb{N}} \subset [0,1)$ *is equidistributed if and only if for any* $k \in \mathbb{Z} \setminus \{0\}$*,*

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k\xi_n} \to 0, \quad \text{as } N \to \infty.$$

From Weyl's criterion, we have following corollaries.

Corollary 1. *Let* $P(x) : \mathbb{R} \to \mathbb{R}, x \mapsto c_m x^m + \cdots + c_0$*, where* $\{c_0, \ldots, c_m\} \subset \mathbb{R}$ *and one of them is irrational number. Then* $\{\langle P(n)\rangle\}_{n\in\mathbb{N}} \subset [0,1)$ *is equidistributed.*

Proposition 2. 设$\alpha \in \mathbb{R} \setminus \mathbb{Q}$. 则$\{\langle n\alpha\rangle\}_{n\in\mathbb{N}}$ 在$[0,1]$ 中稠密.

证法*1.* By Corollary 1 with $m = 1$, $c_0 = 0$, $c_1 = \alpha$, we find that

$$\{\langle n\alpha\rangle\}_{n\in\mathbb{N}} = \{\langle P(n)\rangle\}_{n\in\mathbb{N}} \subset [0,1)$$

is equidistributed, and hence $\{\langle n\alpha\rangle\}_{n\in\mathbb{N}}$ 在$[0,1]$ 中稠密. □

证法2(来自lzc). 欲证$\{\langle n\alpha\rangle\}_{n\in\mathbb{N}}$ 在$[0,1]$ 中稠密, 即证对任意$\varepsilon \in (0,1)$ 和$x \in [0,1]$, 存在$n \in \mathbb{N}$ 使得$|\langle n\alpha\rangle - x| < \varepsilon$. 固定$k \in \mathbb{N}$ 和$x \in [0,1]$, 下面来构造n. 因为$\alpha \in \mathbb{R} \setminus \mathbb{Q}$, 故$\{\langle n\alpha\rangle\}_{n\in\mathbb{N}}$ 两两不等, 从而存在$n, m \in \mathbb{N}$ 使得

$$\langle n\alpha\rangle - \langle m\alpha\rangle \in (0,\varepsilon).$$

注意到

$$na = [n\alpha] + \langle n\alpha\rangle$$
$$ma = [m\alpha] + \langle m\alpha\rangle$$

故

$$\langle (n-m)a\rangle = \langle n\alpha\rangle - \langle m\alpha\rangle \in (0,\varepsilon).$$

取$p := [x/\langle (n-m)a\rangle]$, 则

$$\langle p(n-m)a\rangle = p\langle (n-m)a\rangle \in (x - \langle (n-m)a\rangle, x]$$

故

$$\langle p(n-m)a\rangle - x \in (-\langle (n-m)a\rangle, 0] \subset (-\varepsilon, 0]$$

□

Corollary 2. $\{\sin n\}_{n\in\mathbb{N}}$ *is dense in* $[0,1]$.

References

[1] E. M. Stein and R. Shakarchi, Fourier Analysis: An Introduction, Princeton University Press, 2011.